

On a generalized gravitational Aharonov-Bohm effect

Geusa de A. Marques^{1,2*} and Valdir B. Bezerra^{1†}.

¹*Departamento de Física, Universidade Federal da Paraíba,
Caixa Postal 5008, João Pessoa, Pb, Brazil.*

²*Departamento de Física, Universidade Estadual da Paraíba,
Av. Juvêncio Arruda S/N, Campina Grande, Pb, Brazil.*

Abstract

A massless spinor particle is considered in the background gravitational field due to a rotating massive body. In the weak field approximation it is shown that the solution of the Weyl equations depend on the angular momentum of the rotating body, which does not affect the curvature in this approximation. This result may be looked upon as a generalization of the gravitational Aharonov-Bohm effect.

PACS numbers: 04.20.-q, 0.4.62.+v, 0.4.90.+e

In classical electromagnetism, a charged particle is influenced only by the electromagnetic fields at the location of the particle. There is no physical influence upon a charged particle when it passes through a field-free region.

*gmarques@fisica.ufpb.br

†valdir@fisica.ufpb.br

At the quantum level, however, the behavior of a charged particle is modified by the action of electromagnetic field confined to a region from which the particle is excluded. This nonlocal phenomenon in which electrons are physically influenced by electromagnetic fields not experienced by them is the well-known electromagnetic Aharonov-Bohm effect [1]. This can be understood as a manifestation of a locally trivial vector potential of a gauge field which can lead to observable effects in a nontrivial topology through the phase of the wave function of a charged particle. The electromagnetic Aharonov-Bohm effect [1] has been widely studied theoretically [2], and confirmed experimentally in recent years [3].

In a metric theory of gravitation, a gravitational field is frequently related to a nonvanishing Riemann curvature tensor. However, the presence of localized curvature can have effects on geodesic motion and parallel transport in regions where the curvature vanishes. The best known example of this nonlocal effect is provided when a particle is transported around an idealized cosmic string along a closed curve in which case the string is noticed at all. This situation corresponds to the gravitational analogue [4] of the electromagnetic Aharonov-Bohm effect [1]. These effects are of topological origin rather than local. The electromagnetic Aharonov-Bohm effect represents a global anholonomy associated with the electromagnetic gauge potentials. Its gravitational counterpart may be viewed as a manifestation of nontrivial topology of spacetime. It is worth to call attention to the fact that differently from the electromagnetic Aharonov-Bohm effect which is essentially a quantum effect, the gravitational analogue appears also at a purely classical context. Thus, in summary, the gravitational analogue of the electromagnetic Aharonov-Bohm effect is the following: particles constrained to move in a region where the Riemann curvature tensor vanishes may exhibit a gravitational effect arising from a region of nonzero curvature from which they are excluded. This effect may be viewed as a manifestation of the nontrivial topology of spacetime. In a more general sense, particles con-

strained to move in a region where the Riemann curvature is nonzero, but does not depend on certain parameters such as velocity, like in the case of moving mass currents [5], or the angular momentum of a rotating body [6], in both examples in the weak field approximation, may exhibit gravitational effects associated with each one of these parameters in the respective cases. This kind of gravitational effect we are calling generalized gravitational Aharonov-Bohm effect.

The existence of a gravitational analogue of the electromagnetic Aharonov-Bohm effect was first pointed out by in the end of the sixties of the last century[7-12]. Studies concerning this subject has also received attention by many authors from that time up to now [13].

Effects analogous to the electromagnetic Aharonov-Bohm effect exist in the classical context like the Sagnac effect in general relativity [14] which consists of a phase shift between two beams of light traversing in opposite directions the same path around a rotating mass distribution. With regard to the quantum theory we can mention the experiments using a neutron interferometer to measure the Newtonian gravitational effects on the phase difference of two neutron beams[15,16].

In this paper we consider a massless spinor particle in the spacetime of a rotating massive body and study the influence of the angular momentum of a rotating source on the behaviour of a particle. Gravitational effects are taken into account in the weak field approximation in which case the curvature does not depend on the angular momentum of the source. The generalized gravitational analogue of the electromagnetic Aharonov-Bohm effect set up is this scenario. It is show that in this case there is a nonlocal effect of the angular momentum which is coded in the solution of the Weyl equations.

To begin with let us write down the covariant Dirac equation in a curved spacetime, for a massless spinor field Ψ , which is given by

$$[i\gamma^\mu(x)\partial_\mu + i\gamma^\mu(x)\Gamma_\mu(x)]\Psi(x) = 0, \quad (1)$$

where $\gamma^\mu(x)$ are the generalized Dirac matrices and are given in terms of the standard flat space Dirac matrices $\gamma^{(a)}$ as

$$\gamma^\mu(x) = e_{(a)}^\mu(x)\gamma^{(a)} \quad (2)$$

where $e_{(a)}^\mu(x)$ are tetrad components defined by

$$e_{(a)}^\mu e_{(b)}^\nu \eta^{(a)(b)} = g^{\mu\nu}. \quad (3)$$

Here μ, ν are curved global spacetime indices and a, b are flat tetradic spacetime indices.

The product $\gamma^\mu\Gamma_\mu$ that appears in Eq. (1) can be written as [17]

$$\gamma^\mu(x)\Gamma_\mu(x) = \gamma^{(a)}\left(A_{(a)}(x) + i\gamma^{(5)}B(x)\right), \quad (4)$$

with $\gamma^{(5)} = i\gamma^{(0)}\gamma^{(1)}\gamma^{(2)}\gamma^{(3)}$ and $A_{(a)}$ and $B_{(a)}$ given by

$$A_{(a)} = \frac{1}{2}\left(\partial_\mu e_{(a)}^\mu + e_{(a)}^\rho \Gamma_{\rho\mu}^\mu\right) \quad (5)$$

and

$$B_{(a)} = \frac{1}{2}\epsilon_{(a)(b)(c)(d)}e^{(b)\mu}e^{(c)\nu}\partial_\mu e_\nu^{(d)}, \quad (6)$$

where $\epsilon_{(a)(b)(c)(d)}$ is the completely antisymmetric fourth-order unit tensor.

Now, let us consider the spacetime generated by an infinitely long, infinitely thin massive cylindrical shell rotating around its axis. In the weak field approximation the metric reads [6]

$$ds^2 = -\left(1 - \frac{a(\rho)}{2}\right)dt^2 + \left(1 + \frac{a(\rho)}{2}\right)(d\rho^2 + \rho^2 d\varphi^2 + dz^2) + 2bdt d\varphi, \quad (7)$$

where

$$a(\rho) = -8\mu\Theta(\rho - \rho_0)\ln\left(\frac{\rho}{\rho_0}\right) \quad (8)$$

and

$$b(\rho) = 4\mu w \rho_0 \left[\frac{\rho^2}{\rho_0^2} \Theta(\rho_0 - \rho) + (\rho - \rho_0) \right]. \quad (9)$$

The metric given by Eq. (7) is characterized by two parameters, namely, the linear mass density μ and the linear angular momentum density $j = \mu w \rho_0$. This approximate solution is justified in a domain in which the Newtonian potential generated by the thin massive cylindrical shell is much less than the unity which means that $|a(\rho)| \ll 1$. The term with $b(\rho)$, in this metric, being proportional to j is completely due to the rotation of the shell.

It is interesting to call attention to the fact that, in the weak field approximation, the Riemann curvature tensor outside the rotating shell is completely determined by the function $a(\rho)$ only. The contribution of the term with $b(\rho)$ is concentrated on the shell itself. This means that, in the weak field approximation, the local effects of curvature connected with the rotation of the shell are absent outside it.

In order to solve the Dirac equation for a massless particle, given by Eq. (1), in the spacetime of a rotating massive body given by the line element (7), we will choose the following set of tetrads

$$\begin{aligned} e_{(0)}^\mu &= \left(1 + \frac{a}{4}\right) \delta_0^\mu; & e_{(1)}^\mu &= \left(1 - \frac{a}{2}\right) \delta_1^\mu; \\ e_{(2)}^\mu &= -\frac{b}{\rho} \delta_0^\mu + \frac{1}{\rho} \left(1 - \frac{a}{2}\right) \delta_2^\mu; & e_{(3)}^\mu &= \left(1 - \frac{a}{2}\right) \delta_3^\mu. \end{aligned} \quad (10)$$

Computing the expression for $A_{(a)}$ and $B_{(a)}$ we get

$$A_{(a)} = \left[-\frac{1}{4} \frac{da(\rho)}{d\rho} + \frac{1}{2\rho} \left(1 - \frac{a(\rho)}{2}\right) \right] \delta_{(a)}^1 \quad (11)$$

and

$$B_{(a)} = \frac{1}{4} \frac{b}{\rho^2} \delta_{(a)}^3 \quad (12)$$

As we are considering only the gravitational field in the weak approximation, it is not necessary to distinguish between global and tetradic indices, so that the massless Dirac equation can be written as

$$i\gamma^\mu(x) \left(\partial_\mu + A_\mu(x) + i\gamma^{(5)} B_\mu(x) \right) \Psi(x) = 0. \quad (13)$$

Let us try a solution of Eq. (13) of the form

$$\Psi(x) = \left\{ \exp \left[- \int_c^x dx'^\mu \left(A_\mu(x') + i\gamma^{(5)} B_\mu(x') \right) \right] \right\} \Psi_0(x) \quad (14)$$

where \int_c^x represents an integral along a path c from a given point P up to the endpoint x . Thus, we get the following equation for $\Psi_0(x)$

$$\partial_0 \Psi_0 + \left(1 - \frac{3a}{4} \right) \gamma^\rho \partial_\rho \Psi_0 + \frac{1}{\rho} \left[\left(1 - \frac{3a}{4} \right) - b \right] \gamma^\varphi \partial_\varphi \Psi_0 + \left(1 - \frac{3a}{4} \right) \gamma^z \partial_z \Psi_0 = 0, \quad (15)$$

where γ^ρ and γ^φ are given by

$$\begin{aligned} \gamma^\rho &= \cos \varphi \gamma^1 + \sin \varphi \gamma^2, \\ \gamma^\varphi &= -\sin \varphi \gamma^1 + \cos \varphi \gamma^2. \end{aligned} \quad (16)$$

Now, let us assume that the condition

$$\left(1 + \gamma^5 \right) \Psi_0 = 0 \quad (17)$$

is fulfilled. Equation (17) together with Eq. (1) constitute the Weyl equations for a massless spin 1/2 particle. Thus, our four-dimensional problem reduces to a bidimensional one. Thus, choosing the solution of Eq. (15) as

$$\Psi_0 = \left\{ \exp \left[\left(-iEt + ikz + i \left(n + \frac{1}{2} \right) \varphi \right) \right] \right\} \begin{pmatrix} u_1(\rho) \\ u_2(\rho) \end{pmatrix}, \quad (18)$$

we obtain the following equations for $u_1(\rho)$ and $u_2(\rho)$:

$$\begin{aligned} \frac{d^2 u_1(\rho)}{d\rho^2} + \frac{1}{\rho^2} \left[\frac{3ka}{4(E-k)} \left(n + \frac{1}{2} \right)^2 - \frac{3ka}{4(E+k)} \left(n + \frac{1}{2} \right) \right. \\ \left. - \left(n + \frac{1}{2} \right)^2 (1-a-2b) + \left(n + \frac{1}{2} \right) \left(1 - \frac{5a}{4} - b \right) - \frac{3a}{2} \left(n^2 - \frac{1}{4} \right) \right. \\ \left. + (k-E) \left(1 + \frac{3ka}{4(E-k)} \right) + \frac{3a}{2} \left(\frac{k}{2} - E \right) \right] u_1(\rho) = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{d^2 u_2(\rho)}{d\rho^2} + \frac{1}{\rho^2} \left[\frac{3ka}{4(E-k)} \left(n + \frac{1}{2} \right)^2 + \frac{3ka}{4(E+k)} \left(n + \frac{1}{2} \right) \right. \\ \left. - \left(n + \frac{1}{2} \right)^2 (1-a-2b) - \left(n + \frac{1}{2} \right) \left(1 - \frac{5a}{4} - b \right) + \frac{3a}{2} \left(n + \frac{1}{2} \right) \left(n + \frac{3}{2} \right) \right. \\ \left. + (k-E) \left(1 + \frac{3ka}{4(E-k)} \right) + \frac{3a}{2} \left(\frac{k}{2} - E \right) \right] u_2(\rho) = 0, \end{aligned} \quad (20)$$

whose solutions are given by

$$u_1(\rho) = C_1 \sqrt{\rho} J_{\frac{1}{2}|\nu|} \left(\sqrt{\frac{D}{A}} \rho \right) + C_2 \sqrt{\rho} N_{\frac{1}{2}|\nu|} \left(\sqrt{\frac{D}{A}} \rho \right) \quad (21)$$

$$u_2(\rho) = D_1 \sqrt{\rho} J_{\frac{1}{2}|\mu|} \left(\sqrt{\frac{D}{A}} \rho \right) + D_2 \sqrt{\rho} N_{\frac{1}{2}|\mu|} \left(\sqrt{\frac{D}{A}} \rho \right) \quad (22)$$

where C_1 , C_2 , D_1 and D_2 are normalization constants, $\nu = \frac{\sqrt{(A+4B)A}}{A}$, $\mu = \frac{\sqrt{(A+4C)A}}{A}$, with A , B and C given by

$$A = (E - k)^{-1} \left[\frac{3ka}{4(E - k)} - \left(1 - \frac{3a}{2} \right) \right], \quad (23)$$

$$B = (E - k)^{-1} \left[\left(n + \frac{1}{2} \right)^2 (1 - a - 2b) + \left(n + \frac{1}{2} \right) \left(1 - \frac{5a}{4} - b \right) - \frac{3ka}{4(E - k)} \left(n^2 - \frac{1}{4} \right) \right]. \quad (24)$$

$$C = (E - k)^{-1} \left[\left(n + \frac{1}{2} \right)^2 (1 - a - 2b) + \left(n + \frac{1}{2} \right) \left(1 - \frac{5a}{4} - b \right) - \frac{3ka}{4(E - k)} \left(n + \frac{1}{2} \right) \left(n + \frac{3}{2} \right) \right], \quad (25)$$

The constant D which appears in the argument of the Bessel functions is given by

$$D = k \left(1 - \frac{3a}{4} \right) - E. \quad (26)$$

From the exponential factor in (14) we conclude that $\exp[-\int_c^x dx'^\mu A_\mu(x')]$ gives a contribution independent of the parameter b . On the other hand the factor $\exp[-i\int_c^x \gamma^{(5)} B_\mu(x') dx'^\mu]$ gives a contribution $\exp(-i\beta\gamma^{(5)})$ where $\beta = \int_c^x B_\mu(x') dx'^\mu$ and depends on the angular momentum of the source. In this case if one consider a circularly polarized wave, the effect of this exponential factor is to shift the phase of this wave by an angle $\pm\beta$. If rather, one consider a plane polarized wave, then the effect of the exponential factor $\exp(-i\beta\gamma^{(5)})$ is to rotate the plane of polarization through an angle -2β .

Finally, if we consider the difference in phase or polarization between two beams emitted by a source which follow two different paths in such a way that they form a closed loop around the source, then the relative phase shift depends on the angular momentum of the source.

From the solutions given by Eqs. (21) and (22) we conclude that there is also a dependence on the parameter b . In summary the solution of the Weyl equations depends on the angular momentum of the cylinder through the phase factor of Eq. (14) and the solution of Eq. (15).

It is worth to call attention to the fact that in the region of motion of the massless spin- $\frac{1}{2}$ particle, the Riemann curvature tensor, in the linear approximation, does not depend on the angular momentum of the cylinder of matter, but the solution is influenced by this quantity. This result put into evidence a physical effect, namely, a generalization of the gravitational Aharonov-Bohm effect, which means that even in the case in which the particle is constrained to move in a region where the Riemann curvature does not depend on the angular momentum of the source, it exhibits a gravitational effect associated with this quantity.

REFERENCES

- [1] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
- [2] M. Peshkin and A. Tonomura, *The Aharonov-Bohm effect*, Lecture Notes in Physics, vol. 340, Springer-Verlag, Berlin (1989).
- [3] A. Tonomura, H. Umezaki, T. Matsuda, N. Osakade, J. Eudo and Y. Sugita, Phys. Rev. Lett. **51**, 131 (1983); A. Tonomura, N. Osakade, T. Matsuda, T. Kawasaki, J. Eudo, S. Yano and H. Yamada, Phys. Rev. Lett. **56**, 792 (1986).
- [4] L.H. Ford and A. Vilenkin, J. Phys. **A14**, 2353 (1981); V. B. Bezerra, Phys. Rev. **D35**, 2031 (1987); id. Ann. Phys. (NY) **203**, 392 (1990).
- [5] J. K. Lawrence, D. Leiter and G. Szamosi, Nuovo Cimento **17B**, 113 (1973).
- [6] V. P. Frolov and V. D. Skarzhinsky, Nuovo Cimento **99B**, 67 (1987).
- [7] J. S. Dowker, Nuovo Cimento **52B**, 129 (1967).
- [8] J. S. Dowker and J. A. Roche, Proc. Phys. Soc. **92**, 1 (1967).
- [9] G. Papini, Nuovo Cimento **52B**, 136 (1967).
- [10] D. Wisnivesky and Y. Aharonov, Ann. Phys. (NY) **45**, 479 (1967).
- [11] D. Greenberger, Ann. Phys. (NY) **47**, 116 (1968).
- [12] K. Krauss, Ann. Phys. (NY) **50**, 102 (1968).
- [13] J. L. Safko and L. Witten, Phys. Rev. **D5**, 293 (1972); J. Anandan, Phys. Rev. **D15**, 1448 (1977), id. Nuovo Cimento **53A**, 221 (1979) ; J. Stachel, Phys. Rev. **D26**, 1281 (1982); C. J. C. Burges, Phys. Rev. **D32**, 504 (1985); V. B. Bezerra, J. Math. Phys. **30**, 2895 (1989); B. Jensen and J. Kucéra, J. Math. Phys. **34**, 4975; Vu. B. Ho and M. J. Morgan, Aust. J. Phys. **47**, 245 (1994); M. Alvarez, J.Phys. **A32**, 4079 (1999).
- [14] E. J. Post, Rev. Mod. Phys. **39**, 475 (1967).
- [15] A. Ashtekar and A. Magnon, J. Math. Phys. **16**, 341 (1975); J. Anandan, Phys. Rev. **D15**, 1448

(1977).

- [16] A. W. Overhauser and R. Collela, Phys. Rev. Lett. **33**, 1237 (1974); J. L. Standenmann, S. A. Werner and A. W. Overhauser, Phys. Rev. **A21**, 1419 (1980).
- [17] C. G. Oliveira and J. Tionno, Nuovo Cimento **24**, 672 (1962).